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Topological Properties of Invariant Sets for Anosov Maps with Holes

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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December 2011

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Abstract

Topological Properties of Invariant Sets for Anosov Maps with Holes

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We begin by studying various topological properties of invariant sets of hyperbolic toral automorphisms in the linear case. Results related to cardinality, local maximality, entropy, and dimension are presented. Where possible, we extend the results to the case of hyperbolic toral automorphisms in higher dimensions, and further to general Anosov maps.

Keywords: Dynamical Systems, Open Systems, Hyperbolic Toral Automorphism, Hausdorff Dimension, Box Dimension, Anosov Map

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Acknowledgments

This work would not have been possible without the diligence, patience, and encouragement offered by my advisor, Dr. Todd Fisher. I am also deeply indebted to my parents who have always pushed me to do my best.

Fear thou not; for I am with thee: be not dismayed; for I am thy God: I will strengthen thee; yea, I will help thee; yea, I will uphold thee with the right hand of my righteousness. -Isaiah 41:10

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Chapter 1. Introduction

A relatively new field of dynamical systems is the idea of an open system. An open system presents the possibility that the forward (or reverse) orbit of a given point may "escape" and never return, contributing nothing to the overall dynamics of the system. Aside from pure mathematical intrigue, these systems lend themselves very naturally to certain physical phenomena which are of interest to physical scientists. Prominent examples of application in physics and chemisty include the escaping of a gas from a container (as in [1], [2], and [3]) or modeling sub-atomic quantum phenomena (as in [4] and [5]). Applications from other fields such as ecology have even been considered (see [6]).

In this paper, we further the work done in open systems. We begin by studying invariant sets of open systems formed on tori with the action induced by area-preserving integer-valued matrices. We give results related to cardinality, local maximality, connectedness, entropy and Hausdorff dimension. Whenever possible, we extend the results to more general Anosov maps.

Billiards present a common example in the study of open systems. In a dynamical billiard, one or more particles travel at unit speed along straight-line trajectories within sets $X \subset \mathbb{R}^2$, with angle-reflecting and velocity-preserving collisions. For each $(x, \theta) \in X \times \mathbb{S}^1$, there is an orbit passing through x with angle θ. Behaviors of the system are categorized in terms of these points. For instance, (x, θ) is periodic if the orbit passing through x at angle θ eventually returns to x, moving in the same direction. Billiards are divided into two categories, inner and outer billiards. A dynamical billiard is an inner billiard if X is bounded, and is an outer billiard if X is unbounded. Open systems can be considered in

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dynamical billiards by considering only the points whose orbit stay bounded in the case of an outer billiard. For inner billiards, a portion of the boundary can be removed, and the orbits which fall into the removed portion of the boundary are considered to have escaped.

One example of the latter case is given in [1]. In the paper, Alt et. al. consider the orbits of particles in the Bunimovich stadium billiard (a rectangle with two congruent semicircles attached to opposite ends, resembling a medicinal capsule), with a small hole cut in one of the boundaries. The paper gives numerical estimates for the probability that the orbit of an arbitrarily chosen particle will remain in the stadium for a given length of time. The decay is found to be roughly exponential as a function of the length of the orbit. Further work on the same problem is done by Dettmann and Georgiou in [2], where explicit values for the constant C in the decay equation Ct^{-1} are computed in terms of the size of the hole.

These ideas are furthered in [3], where Demers, Wright, and Young consider arbitrary holes both on the boundary and in the interior of arbitrary billiards. They are able to prove that for a certain class of initial distributions, there is a common escape rate as well as a common limiting distribution. They also show that as the sizes of the holes tend to zero, the distributions, thought of as measures, tend to the natural invariant measure of the corresponding billiard without holes.

An outer billiard example is given in [7]. Lopes and Markarian consider an open billiard system in which three circular scatterers are placed at the vertices of an equilateral triangle. The authors are able to determine that the set of all points whose forward and reverse orbits bounce of the scatterers infinitely many times for forward and reverse time form a Cantor set, and the iteration map restricted to this invariant set is similar to Smale's horseshoe map.

This last example hints at a very natural question regarding this type of dynamical sys-

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tem: “What’s left?” The set of points in an open dynamical system which never leave the system form an invariant set. Certain properties of such invariant sets are known. For instance, due to work by Xia in [8] and by Bochi and Viana in [9], measure theoretic properties of invariant sets are considered. A key result of this work is that for any volume-preserving diffeomorphism of a manifold, if a closed, invariant set is hyperbolic and has non-empty interior, it must be the entire manifold, and the diffeomorphism is therefore Anosov. The converse of this statement, therefore, applies to open systems: For any volume-preserving Anosov diffeomorphism of a manifold, any invariant set which is not the whole manifold has empty interior.

In [10], Bundfuss et. al. study a purely dynamical example of an open system. They consider the unit interval with a non-invertible interval map, and remove a finite number of connected open intervals from the unit interval. The topological and dynamical structures of the resulting invariant set are studied. In particular, the authors are able to provide some interesting results on the cardinality of topologically transitive components as well as an upper bound on the number of possible topologically transitive components in terms of the number of holes removed.

Another property that can be used in describing invariant sets is dimension. This too has been studied in manners that are immediately applicable to open systems. For instance, in [11], Urbański considers the set of points of a compact Riemannian manifold M. It is shown that the set of all points of M whose full orbits are not dense in M have the same Hausdorff dimension as M itself.

A particularly interesting study in dimension is given by Przytycki in [12]. In this paper, Przytycki shows that for the n-dimensional torus, there exist invariant subsets of *topological* dimension $1, 2, ..., n-2$, and that it is not possible to construct an invariant subset of

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dimension $n-1$ on the torus.

A direct application of dimension studies to open systems is given in a two-part paper by Horita and Viana (see [13] and [14]). In the first, the authors consider the invariant set of a piecewise-smooth map on a manifold with holes. They are able to show that the Hausdorff dimension of the repeller is less than the Hausdorff dimension of the ambient manifold. The study is continued in the second paper, where they consider transitive Anosov diffeomorphisms through Hopf bifurcations.

In the next chapter, we will present necessary definitions and background information. Chapter 3 will give some basic results. Chapter 4 will focus on results related to topological entropy and the dimension of invariant sets.

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2.1 Basics

A *dynamical system* (X, f) is a topological space X together with a continuous map $f : X \to X$. Among the goals of the study of dynamical systems is to study/characterize the long-term behaviors of subsets of the set X under repeated iteration of the map f. If f is a homeomorphism, the study of subsets of X under iterations of f^{-1} are also considered. Some simple behaviors are those of *fixed points*, *periodic points* and *invariant sets*:

- A point $x \in X$ is said to be a *fixed point* if $f(x) = x$.
- A point $x \in X$ is said to be *periodic* if there is some $n \in \mathbb{N}$ such that $f^n(x) = x$. If n is the smallest positive integer such that $f^n(x) = x$, then n is said to be the *minimal period* of the point x.
- A set $\Lambda \subset X$ is said to be *invariant* if $f(\Lambda) = \Lambda$.

It is worth clarifying that for an invariant set, the set Λ need not be fixed point-wise. For instance, if x is a periodic point with period n, then the set $\Lambda = \{x, f(x), f^2(x), ..., f^{n-1}(x)\}$ is invariant, but clearly not fixed point-wise.

For a point $x \in X$, the *forward orbit* of x is equal to the set

$$\{f^n(x) \ : \ n \in \mathbb{N} \cup \{0\}\}$$

and is denoted $\mathcal{O}^+(x)$. Similarly, if f is invertible, the *reverse orbit* $\mathcal{O}^-(x)$ is the set

$$\{f^{-n}(x) \ : \ n \in \mathbb{N} \cup \{0\}\}.$$

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Finally, the *orbit* of x is denoted by $\mathcal{O}(x)$ and is equal to $\mathcal{O}^+(x) \cup \mathcal{O}^-(x)$.

A map $f : X \to X$ is said to be *transitive* on an invariant set Λ if the forward orbit of some point $x \in \Lambda$ is dense in Λ, e.g. for any point $y \in \Lambda$, there is a point of $\mathcal{O}^+(x)$ arbitrarily close to y. A map is said to be *topologically mixing* if given two open sets $U, V \subset X$, there is some positive integer n_0 such that for all $n \geq n_0$, $f^n(U) \cap V \neq \emptyset$. Any map which is mixing is automatically transitive.

2.2 Conjugacy

The idea of a *topological conjugacy* is analogous to the idea of a similarity transformation in matrix theory or an isomorphism in algebra. Two maps $f : X \to X$ and $g : Y \to Y$ are said to be *topologically conjugate* if there is a homeomorphism $k : X \to Y$ that satisfies $k \circ f = g \circ k$, or (re-arranging) $f = k^{-1} \circ g \circ k$. In this case, k is said to be a *topological conjugacy* between f and g. Topological conjugacy preserves topological properties such as topological entropy, but does not necessarily preserve properties such as Hausdorff dimension (both of which will be addressed later). Sometimes it is easier to study behavior in the conjugate system rather than in the original setting.

A weaker but related idea is that of a *topological semi-conjugacy.* If $f : X \to X$ and $g : Y \to Y$ satisfy $k \circ f = g \circ k$ for some continuous surjective map k, not necessarily a homeomorphism, then k is said to be a *topological semi-conjugacy.* As can be expected, certain properties are not preserved under semi-conjugacy, but often it is the case that some information can be recovered in the form of inequalities. For instance, if both X and Y are compact, then $h_{\text{top}}(f) \geq h_{\text{top}}(g)$, where $h_{\text{top}}(f)$ and $h_{\text{top}}(g)$ represent the topological entropy of the maps f and g, respectively. (See [15], p. 376)

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2.3 Symbolic Dynamics

Let $\Sigma_n^+ = \{s_0s_1s_2s_3s_4... : s_i = 1, 2, ..., n\}$ be the space of one-sided infinite sequences on n symbols. Let $\sigma : \Sigma_n^+ \to \Sigma_n^+$ be the map defined by

$$\sigma(s_0s_1s_2s_3s_4...) = s_1s_2s_3s_4$$

e.g. the map σ simply "pulls off" the first symbol in the sequence. Here, σ is called the *shift map*, and (Σ_n^+, σ) is called the *full one-sided n-shift space*. This space also has a natural topology. For a fixed $m \in \mathbb{N}$, a *cylinder set* is the set of all sequences whose first m entries are equal. The collection of all cylinder sets forms the basis of topology on Σ_n^+.

An extension is to consider the space of bi-infinite sequences on n symbols. In this case, we let $\Sigma_n = \{...s_{-2}s_{-1}.s_0s_1s_2... : s_i = 1, 2, ..., n\}$, and the map σ is now defined as

$$\sigma(...s_{-2}s_{-1}.s_0s_1s_2...) = s_{-2}s_{-1}s_0.s_1s_2$$

with the period in the middle of the sequence denoting the "current" position in the sequence. A key difference between the space (Σ_n^+, σ) and (Σ_n, σ) is the fact that σ is invertible in the second case and n-to-one in the first. (Σ_n, σ) is called the *full n-shift space*. Cylinder sets can also be defined in this case. In particular, the set of all sequences whose entries between $-m$ and m are equal form a cylinder set.

One further extension involves an $n \times n$ matrix A, with $a_{ij} = 0, 1$ for each entry of the matrix. (We assume that each row and column of A have at least one non-zero entry.) In this case, we define $\Sigma_A^+ = \{s_0s_1s_2s_3 : s_i = 1, 2, 3, ..., n, a_{s_is_{i+1}} = 1\}$. That is to say, we only allow sequences where consecutive symbols are "adjacent" in terms of the matrix A. The space Σ_A is defined in an analogous fashion. The spaces (Σ_A^+, σ) and (Σ_A, σ) are called *subshifts*

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of finite type.

Each of the three preceding are examples of *symbolic dynamics*. These types of spaces are natural targets for topological conjugacy, as the behavior on shift spaces or subshifts of finite type are often far more easily understood than their conjugate counterparts.

2.4 Hyperbolic Dynamics

Let M be a manifold, and $f : M \to M$ be a C^1 diffeomorphism. A periodic point x with minimal period m is said to be a *hyperbolic periodic point* if there is a splitting of the tangent space $T_xM = \mathbb{E}^u_p \bigoplus \mathbb{E}^s_p$ so that $\mathbb{E}^u_p$ is expanding under the map $Df^m(p)$ and $\mathbb{E}^s_p$ is contracting under the map $Df^m(p)$. (Recall that in applying the map $Df^m(p)$ to the subspaces, we identify the point p with the origin in $\mathbb{R}^l$, where l represents the dimension of the subspace.)

Extending this notation, let Λ be any invariant set of the same map f. The set Λ has a *uniform hyperbolic structure* if for every point $p \in \Lambda$ (not necessarily periodic), there is a splitting of $T_pM = \mathbb{E}^u_p \bigoplus \mathbb{E}^s_p$ satisfying:

- $Df(p)(\mathbb{E}^u_p) = \mathbb{E}^u_{f(p)}$, $Df(p)(\mathbb{E}^s_p) = \mathbb{E}^s_{f(p)}$
- There are constants $0 < \lambda < 1$ and $C \geq 1$, not depending on the point $p \in \Lambda$, such that for all $m \geq 0$ we have:
  - $|Df^n(p)v^s| \leq C\lambda^n|v^s|$ for all $v^s \in \mathbb{E}^s_p$, and
  - $|Df^{-n}(p)v^u| \leq C\lambda^n|v^u|$ for all $v^u \in \mathbb{E}^u_p$.

A compact invariant subset of a manifold M having a uniform hyperbolic structure is appropriately called a *hyperbolic invariant set.* If $\Lambda = M$, then the map f is said to be an *Anosov*

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diffeomorphism.

A stronger condition than hyperbolicity is that of *conformality.* A map f is said to be *conformal* if df_x is a scalar multiple of an isometry on T_xM. The scalar multiple is allowed to vary with the point x. (See [16], p. 199.) f is said to be *u-conformal* (respectively *s-conformal*) if $f|_{\mathbb{E}^u}$ $(f|_{\mathbb{E}^s})$ is conformal. (See [16], p. 230.)

If p is a hyperbolic fixed point for a C^k map f and U is a neighborhood of p, we define the *local stable manifold* of p, denoted $W^s_U(p)$, to be the set of all points x in U such that the distance between $f^n(x)$ and p approaches 0 as $n \to \infty$. Similarly, the local unstable manifold of p, $W^u_U(p)$, is the set of all points x in U such that the distance between $f^{-n}(x)$ and p approaches 0 as $n \to \infty$ in the case that f is invertible. In the case that f is not invertible, x belongs to $W^u_U(p)$ if it is possible to create a sequence $\{x_i\}$ with each x_i chosen to be one point of $f^{-i}(x)$ such that $\{x_i\}$ converges to p.

Both of $W^s_U(p)$ and $W^u_U(p)$ can be extended to form the *global stable manifold* and *global unstable manifold* of p, denoted $W^s(p)$ and $W^u(p)$ respectively. $W^u(p)$ is formed by taking the union of all forward iterates of $W^u_U(p)$ under f. $W^s(p)$ is formed by taking the union of all inverse images of $W^s_U(p)$ under f if f^{-1} exists. In the event that f is not invertible, we form a sequence of sets $W^s_U(p) = X_0 \subset X_1 \subset X_2 ...$ with $x \in X_i$ if $f(x) \in X_{i-1}$. The proof that these sets are actually manifolds can be found in [15], pp. 187-199.

If p is a hyperbolic periodic point, we can perform a similar procedure by replacing f by f^m and replacing f^{-1} by f^{-m}, where m is the period of p. Since p is a fixed point of f^m, all of the above results hold. Further, for a general point $p \in M$, not necessarily periodic, we define

$$W^s(p) = \{x \in M \ : \ d(f^n(x), f^n(p)) \to 0 \text{ as } n \to \infty\}$$

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where d is a complete metric on M. Similarly, if f is invertible, we can define

$$W^u(p) = \{x \in M \ : \ d(f^{-n}(x), f^{-n}(p)) \to 0 \text{ as } n \to \infty\}.$$

If f is not invertible, we can adapt the above definition as follows: x belongs to $W^u_U(p)$ if it is possible to create a sequence $\{x_i\}$ with each x_i chosen to be one point of $f^{-i}(x)$ such that $d(x_i, f^{-i}(p))$ converges to 0 as i tends to infinity.

An important result about $W^s(p)$ and $W^u(p)$ is the stable manifold theorem for a hyperbolic set. This states that if $f : M \to M$ is a C^k diffeomorphism, and Λ is a hyperbolic invariant set for f, then there is some $\epsilon > 0$ such that for all $p \in \Lambda$ there are two C^k embedded disks $W^s_\epsilon(p, f)$ and $W^u_\epsilon(p, f)$ which are tangent to $\mathbb{E}^s_p$ and $\mathbb{E}^u_p$ respectively, on which $W^s_\epsilon(p, f)$ is the graph of a C^k function $\mathbb{E}^s_p \to \mathbb{E}^u_p$, and $W^u_\epsilon(p, f)$ is the graph of a C^k function $\mathbb{E}^u_p \to \mathbb{E}^s_p$. Further, both of these functions and their first k derivatives vary continuously as p varies.

For a hyperbolic periodic point p, a point q lying in the intersection of the stable and unstable manifolds through p is said to be a *homoclinic point* of p. For such a point, $f^n(q)$ approaches $\mathcal{O}(p)$ as $n \to \infty$ and as $n \to -\infty$ (assuming f^{-1} exists). Further, q is called a *transverse homoclinic point* if the vector space sum of the stable and unstable tangent spaces of $W^s(p)$ and $W^u(p)$ at q has full dimension. An important result related to homoclinic points is the *Smale-Birkhoff Theorem* (see [15], p. 288.) . This states that if q is a transverse homoclinic point for a hyperbolic periodic point p of a diffeomorphism f, then for each neighborhood U of the two-point set $\{p, q\}$ there is a positive integer n such that f^n has a hyperbolic invariant set $\Lambda \subset U$ with $p, q \in \Lambda$ and on which f^n is topologically conjugate to the two-sided shift map on Σ_2.

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A useful property for hyperbolic invariant sets to posses is *local maximality*. A hyperbolic invariant set Λ is *locally maximal* (with respect to the map f) if there exists an open set $U \supset \Lambda$ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

In other words, Λ is the largest invariant set contained in a neighborhood of itself. An equivalent condition for local maximality is that of a *local product structure*. The set Λ has a *local product structure* if there exists $\delta > 0$ such that for all points x and y that are less than δ apart, the local stable manifold through x and the local unstable manifold through y intersect in a single point z and $z \in \Lambda$. (See [17], p. 272.)

2.5 Markov Partitions

A *rectangle* is a subset R of a manifold that satisfies $R = \overline{\mathrm{int}(R)}$, e.g. R is the closure of its own interior. In certain cases, the rectangles look like rectangles in the usual geometric sense. An example of such is hyperbolic toral automorphisms of the 2-torus, which will be discussed at length in the following chapters. Markov partitions are a finite collection of rectangles $\{R_i\}$ that partition a hyperbolic manifold with overlap only along the boundaries, and whose construction is closely tied in with the dynamics of the system. Specifically, for an Anosov diffeomorphism on a manifold M, the collection $\{R_i\}$ form a Markov partition if:

1. $M = \bigcup_{i=1}^{n} R_i$
2. the interiors of distinct rectangles are disjoint
3. for any two distinct points x and y in a single R_i, the stable manifold of x lying in R_i and the unstable manifold of y lying in R_i intersect in a single point $z \in R_i$.
4. $f(W^u(x, R_i)) \supset W^u(f(x), R_j)$ where $x \in R_i$, $f(x) \in R_j$, and $W^u(x, R_i)$ denotes the

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part of the unstable manifold of x contained in R_i.

5. $f(W^s(x, R_i)) \subset W^s(f(x), R_j)$ where $x \in R_i$, $f(x) \in R_j$, and $W^s(x, R_i)$ denotes the part of the stable manifold of x contained in R_i.

(See [18].) One algorithm for constructing Markov partitions for the 2-torus is given in [15], p. 313.

Associated with each of these partitions is a *transition matrix* T, which describes the behavior of each of the rectangles under iteration by f. The transition matrix T has as many rows as rectangles in the partition. Each entry t_{ij} of the matrix T is the number of connected components of $f(R_i) \cap R_j$. If the rectangles are sufficiently small, each entry will be either 0 or 1. Markov partitions provide the mechanism for a semi-conjugacy between hyperbolic and symbolic dynamics.

A transition matrix T is said to be *irreducible* if for all $i, j \in \{1, 2, ..., n\}$ there exists $k \in \mathbb{N}$ such that $t^k_{ij} > 0$, where t^k_{ij} represents the ij entry of T^k. If the transition matrix is thought of as an adjacency matrix for a directed graph, this is equivalent to strong connectivity of the graph–i.e. there is a directed path between any two vertices. A transition matrix that is not irreducible is said to be *reducible*.

If there exists some $k \in \mathbb{N}$ such that $T^k_{ij} > 0$ for all i, j, then T is *eventually positive*. Any matrix that is eventually positive is irreducible, but not all irreducible matrices are eventually positive. For instance,

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is irreducible but not eventually positive.

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Remark: Computing the transition matrix associated with the Markov partition is generally done by brute force. For purposes of this paper, a MATLAB program was written to perform this computation in the case of two-dimensional hyperbolic toral automorphisms. The source code has been included as an appendix.

2.6 Entropy

One very important measurement in a dynamical system is that of *topological entropy.* As suggested by its name, this is quantity that reflects the complexity of the system. Given a metric space X with metric d, continuous map $f : X \to X$, and $n \in \mathbb{N}$, define a distance function

$$d_n(x, y) = \max_{0 \leq j < n} (d(f^j(x), f^j(y)).$$

In other words, two points of X are close in the d_n metric if they remain close under n iterations of the map f.

Next, we define an (n, ϵ)-*separated* set. A set $S \subset X$ is said to be (n, ϵ)-*separated* for f if $d_n(x, y) > \epsilon$ for any distinct $x, y \in S$. Define a new function

$$r_{\text{sep}}(n, \epsilon) = \max\{|S| \ : \ S \subset X \text{ is an } (n, \epsilon)\text{-separated set}\},$$

where by $|S|$ we mean the cardinality of the set S. Note that in the case of X being a compact set the value of $r(n, \epsilon)$ will be finite.

Lastly, the entropy of a map f, $h_{\text{top}}(f)$, measures the average rate of exponential growth

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of the size of these (n, ϵ)-separated sets as n and ϵ tend to zero. Specifically,

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \frac{\log(r_{\text{sep}}(n, \epsilon))}{n} \right),$$

where $h_{\text{top}}(f)$ is given the value ∞ if the inner limit is infinite.

Alternatively, a set $S \subset X$ is said to be (n, ϵ)-*spanning* for f if for any point $x \in X$ there is a point $y \in S$ such that $d_n(x, y) \le \epsilon$. Defining $r_{\text{span}}(n, \epsilon)$ to be the smallest number of elements in an (n, ϵ)-spanning set, we can also define the entropy of the system as

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \frac{\log(r_{\text{span}}(n, \epsilon))}{n} \right).$$

A third definition of entropy can be derived in terms of open covers. A collection $\mathcal{A}$ of subsets of X is called an *open cover of* X if each $A \in \mathcal{A}$ is an open subset of X and the union of all $A \in \mathcal{A}$ is equal to X. A subcollection $\mathcal{B}$ is called a *subcover* if the union of all $A \in \mathcal{B}$ is equal to X. Define

$$\mathcal{A}^n = \left\{ \bigcap_{j=0}^{n-1} f^{-j}(A_j) : A_j \in \mathcal{A} \text{ and } \bigcap_{j=0}^{n-1} f^{-j}(A_j) \neq \emptyset \right\}.$$

Let $N(\mathcal{A})$ denote the minimum cardinality of a subcover $\mathcal{B} \subset \mathcal{A}$. Define

$$h(\mathcal{A}, f) = \limsup_{n \to \infty} \frac{\log(N(\mathcal{A}^n))}{n}.$$

The entropy of the system is then given by

$$h_{\text{top}}(f) = \sup\{h(\mathcal{A}, f) \ : \ \mathcal{A} \text{ is an open cover of } X\}.$$

In [15], Lemma 1.10, p. 380, it is demonstrated that all of these definitions of entropy are

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equal.

Another important result on topological entropy is that the entropy of a subshift of finite type with adjacency matrix A is the modulus of the eigenvalue furthest from the origin. (See [15], Theorem 1.9, p. 376.) This also holds in the case of Markov partitions, where the transition matrix need not contain only ones and zeros. Topological entropy is one quantity that is preserved under conjugacy as well as under uniform finite-to-one semi-conjugacy, as in the case of hyperbolic toral automorphism.

For an invariant Borel probability measure μ, the *measure-theoretic entropy* of f is denoted $h_\mu(f)$. Roughly speaking, the measure-theoretic entropy measures the exponential growth rate of orbits which are "relevant" to μ. (A precise definition of measure-theoretic entropy can be found in [17], p. 169.)

For an f-invariant set $\Lambda \subset M$, consider the quantity

$$\sup_{\mu \in \mathfrak{M}(\Lambda)} h_\mu(f)$$

where $\mathfrak{M}(\Lambda)$ is the set of all f-invariant Borel ergodic measures on M. If there exists a measure μ^* for which

$$h_{\mu^*}(f) = \sup_{\mu \in \mathfrak{M}(\Lambda)} h_\mu(f),$$

then the measure μ^* is called a *measure of maximal entropy.* A result that relates measure-theoretic entropy and topological entropy is the *variational principle* (or the *variational principle for entropy*), which states that

$$h_{\text{top}}(f) = \sup_{\mu \in \mathfrak{M}(\Lambda)} h_\mu(f).$$

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(See [17], p. 181.)

Another important concept that is related to topological entropy is that of *topological pressure*. This is another technique for measuring the exponential growth rate of orbits with respect to some weighting function ϕ. If $P_X(\phi)$ denotes the topological pressure of ϕ on a space X, then the *variational principle (for pressure)* is given by

$$P_X(\phi) = \sup_{\mu \in \mathfrak{M}(\Lambda)} h_\mu(f) + \int_X \phi \, d\mu.$$

(See [17], p. 625.) A measure μ^* for which

$$P_X(\phi) = h_{\mu^*}(f) + \int_X \phi \, d\mu^*.$$

is called an *equilibrium state*. More information on these concepts will not be necessary for understanding the results in this thesis. The curious reader is referred to the references indicated for more information.

2.7 Dimension

Multiple definitions exist for computing the dimension of a set. We focus here on two commonly used definitions: box dimension and Hausdorff dimension.

For a compact set $S \subset \mathbb{R}^n$, consider a grid of n-dimensional cubes of side length ϵ, which overlap only on their common boundaries. Let $N(\epsilon, S)$ be the number of cubes that contain at least one point of S. The *lower box dimension* (or *inner box dimension*) of S is defined as

$$\dim_b(S) = \liminf_{\epsilon \to 0} \frac{\log(N(\epsilon, S))}{-\log(\epsilon)}$$

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and the *upper box dimension* (or *outer box dimension*) of S is defined as

$$\dim_B(S) = \limsup_{\epsilon \to 0} \frac{\log(N(\epsilon, S))}{-\log(\epsilon)}.$$

When the two coincide, the limit is the *box dimension* of the set S.

It is also possible to define the box dimension by using a related quantity. If we define $N'(\epsilon, S)$ to be the minimum number of ϵ-cubes covering S (not necessarily in a grid), replacing N by N' in the definitions above will yield the same number.

Alternatively, we can consider the Hausdorff dimension of a compact set S. First, we define the *d-dimensional Hausdorff Content* of S, denoted $C_H^d(S)$. This is given by

$$C_H^d(S) = \inf \left\{ \sum_i r_i^d \mid S \subset \bigcup_i B(x_i, r_i) \right\}$$

where the infimum is taken over all collections of balls covering S.

The *Hausdorff Dimension* of a set S, $\dim_H(S)$, is given to be the infimum of all non-negative numbers d such that the d-dimensional Hausdorff content of S is zero.

From the definitions, the following are also apparent:

- If $A \subset B$, then $\dim_H(A) \leq \dim_H(B)$.
- $\dim_H(A \cup B) = \max\{\dim_H(A), \dim_H(B)\}$.

The above statements remain true if the subscript H is replaced by either B or b.

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Finally, if $S \subset \mathbb{R}^n$ is compact, we have

$$0 \leq \dim_H(S) \leq \dim_b(S) \leq \dim_B(S) \leq n.$$

An important equation that is related to dimension is *Bowen's equation*. For a negative function ϕ on a set Λ, the function $\psi(t) = P_\Lambda(t\phi)$ has a unique positive root. For properly defined functions ϕ, the root of this equation gives information about the Hausdorff dimension of the set Λ. For instance, in [16], Theorem 22.2, the Hausdorff dimension of a locally maximal hyperbolic set with some additional technical conditions is given as a sum of two roots of Bowen's equation for two functions ϕ_1 and ϕ_2.

Some well-known examples of all three dimensions being equal include the Cantor middle-thirds set, which has all three dimensions equal to $\log_3(2)$, and the Sierpinski triangle, which has all three dimensions equal to $\log_2(3)$. However, the inequalities between all three dimensions may be strict. For example, the set of rational numbers on the unit interval $I = [0, 1]$ has zero Hausdorff dimension (in fact, *any* countable set has zero Hausdorff dimension). But since $\mathbb{Q}$ is dense in I, any box in I will contain a point of $\mathbb{Q}$. Hence, the quantity $N(\epsilon, I \cap \mathbb{Q})$ is equal to the number of boxes needed to cover I, which is roughly $1/\epsilon$. (Strictly speaking, this quantity should be rounded up to the nearest integer, but this additional "noise" vanishes in the limit.)

It is also possible to construct sets for which the lower box dimension and upper box dimensions are not equal. For example, let S be the set of all numbers of the form

$$0.0xx0000xxxxxxxx0000000000000000...$$

where the x represents any digit. That is to say, we consider all numbers in I with a block

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of zeros having length 1, then a block of arbitrary digits having length 2, then a block of zeros having length 4, etc. A simple combinatorial argument can be used to find $N(\epsilon, S)$ for $\epsilon = 10^{-2^n}$ for $n \in \mathbb{N}$. The behavior for these is different depending on whether n is even or odd. Because of this, $\dim_b(S) = 1/3$ and $\dim_B(S) = 2/3$.

These examples where the three dimensions differ lack the structure imposed by a dynamical system. In these cases, it may be true that the Hausdorff dimension and box dimensions are equal. In [16], p. 119, conditions that will give equality are listed. Further, in section 4.1 of [19], it is shown that a certain construction will always give a case where these quantities are all equal. General statements relating to dynamical systems, however, are notably absent.

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As stated in the introduction, the goal of this work will be to give as much information as can be given about a particular class of hyperbolic invariant sets. To approach the problem, we give a few preliminary standing assumptions. Unless otherwise stated, in this and succeeding chapters, A will be understood to be a matrix with determinant ± 1, non-negative integer entries, and no eigenvalue of unit modulus. For many examples, we will let A be 2×2. Further, $f_A : \mathbb{T}^n \to \mathbb{T}^n$ is defined by the map $f_A(\mathbf{x}) = A\mathbf{x} \bmod \mathbb{Z}^n$. Since the determinant of A is ± 1, f_A defines a C^∞ automorphism, and is called a *hyperbolic toral automorphism.* Also, the manifold $\mathbb{T}^n$ is a hyperbolic set for the map f_A. Finally, if $U \subset \mathbb{T}^n$ is open and non-empty, we define

$$\Lambda_U = \{\mathbf{x} \in \mathbb{T}^n : f^n(\mathbf{x}) \notin U, \forall n \in \mathbb{Z}\}.$$

Remark: This notation is not in agreement with standard notation. Indeed, Λ_U generally represents the intersection of all forward and backward images of $\overline{U}$. In this sense, the sets that we call Λ_U would commonly be known as Λ_{U^c}, with c denoting the complement. For convenience, we will use our "adapted" notation to avoid the cumbersome use of the complement in all discussions.

It is not difficult to see that the set Λ_U is an invariant set under the map f, and it is precisely these sets whose properties we wish to describe. Additionally, we will use Λ to denote a general invariant set, and Λ_U to denote a particular invariant set from a hyperbolic toral automorphism.

We start by giving some elementary well-known results that will be important later on in our analysis of the invariant sets. We will then proceed to give a result related to con-

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nectedness as well as two results on the cardinality of the invariant set.

3.1 Preliminary Remarks on Area-preserving Linear Maps

The next two results are well-known, and proofs are included for completeness.

Theorem 3.1. *If A satisfies the conditions listed above and $n = 2$, then:*

- *A has real eigenvalues,*
- *The eigenvalues of A are irrational, and*
- *The ratio of the two entries of both eigenvectors is irrational.*

Proof. Suppose A has a complex eigenvalue z. Then $\overline{z}$ is also an eigenvalue of A. Since the product of the eigenvalues of A is the determinant of A, then $z\overline{z} = |z|^2 = 1$, violating the assumption that no eigenvalue of A has unit modulus.

Suppose the matrix A is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The eigenvalues of A are then the solutions of the quadratic

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

Recognizing $ad - bc$ as the determinant of A, the solutions of this equation are given by

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4\det(A)}}{2}.$$

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Suppose that the eigenvalues of A are rational. Then the expression under the radical must be a perfect square that is exactly four more or four less than some other perfect square. There are only two such numbers, namely 0 and 4. If $\det(A) = 1$ then it must be that $(a+d)^2 = 4$. Then $(a+d) = \pm 2$. This gives

$$\lambda = \frac{\pm 2 \pm \sqrt{4-4}}{2} = \pm 1$$

contrary to assumption. Conversely, if $\det(A) = -1$, then if the eigenvalues are rational $(a+d)^2 = 0$ which again gives a unit-modulus eigenvector by a similar computation.

Let $\mathbf{x}$ be an eigenvector of A, λ its corresponding eigenvalue. Suppose one entry of $\mathbf{x}$ is zero. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda x_2 \end{bmatrix},$$

which gives $dx_2 = \lambda x_2$, so $d = \lambda$. But $\lambda \notin \mathbb{Q}$, $d \in \mathbb{Z}$, an obvious contradiction. Hence, we can scale $\mathbf{x}$ so that the first entry is 1. This gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda x_2 \end{bmatrix}.$$

Suppose $x_2 \in \mathbb{Q}$. Then $a + bx_2 = \lambda$. But $a, b, x_2 \in \mathbb{Q}$ and $\lambda \notin \mathbb{Q}$. Hence $x_2 \notin \mathbb{Q}$, which gives the ratio of $x_1/x_2 \notin \mathbb{Q}$. $\square$

Theorem 3.2. *If f_A is a hyperbolic toral automorphism of dimension n, the set of periodic points of the map f is dense in $\mathbb{T}^n$.*

Proof. Let $\mathbf{x}$ be a column vector of a point with both rational entries. Without loss of generality each entry of $\mathbf{x}$ has common denominator d. Then $f_A(\mathbf{x})$ is also a point of $\mathbb{T}^n$ with all rational entries. Further, these entries again have common denominator d. Since

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there are only finitely many points of $\mathbb{T}^n$ with all rational entries whose denominator is d, successive images of the point $\mathbf{x}$ under f_A must eventually repeat. Hence, the point $\mathbf{x}$ must be periodic. This is true for all points $\mathbb{Q}^n$, so points of $\mathbb{Q}^n$ are periodic. Further, these points are dense in $\mathbb{T}^n$, completing the proof. □

3.2 Connectivity of Invariant Sets

We now give the first topological property about the set Λ_U associated with an open set U. Recall that a set is said to be *totally disconnected* if its only connected components are single points.

Theorem 3.3. *For any non-empty open set $U \subset \mathbb{T}^2$ and hyperbolic toral automorphism f_A, the associated invariant set Λ_U is totally disconnected.*

Proof. Let U be an arbitrary open set in $\mathbb{T}^2$. By Theorem 3.2, we know that U contains a periodic point z. Let L be an open segment of $W^u(z)$ entirely contained in U which contains z. Then, if n is the period of z under f, it will be true that $L \subset f^n(L)$. Further, $f^{mn}(L) \subset f^{(m+1)n}(L)$ for all natural numbers m. Also, since $W^u(z)$ has irrational slope when viewed as a line, the inclusion will always be proper. Then $W^u(z)$ be the union of all images $f^{mn}(L)$ for all natural numbers m. As the slope of $W^u(z)$ is irrational, $W^u(z)$ is dense in $\mathbb{T}^2$. Note that this can be repeated using f^{-1} and $W^s(z)$.

Let x and y be distinct points of Λ_U. Let P_x and P_y be two parallelograms whose sides are segments of $W^u(z)$ and $W^s(z)$ such that $x \in P_x$, $y \in P_y$, and $P_x \cap P_y = \emptyset$. (The last condition is possible by the density of $W^u(z)$ and $W^u(z)$.) Hence there are two disjoint open sets in Λ_U that contain one of x or y but not both, and Λ_U is therefore totally disconnected. □

Remark: As the stable and unstable manifolds through periodic points are also dense for

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Anosov maps on 2-manifolds, this result holds in that case as well.

3.3 Cardinality of Invariant Sets

We now present two theorems that give a relationship between the "size" of the set U and the cardinality of the set Λ_U. As each of these theorems are both true for any transitive Anosov map f on an n-manifold M, we will present the theorems in this context. In this sense, Λ_U will represent

$$\Lambda_U = \{\mathbf{x} \in M : f^n(\mathbf{x}) \notin U, \forall n \in \mathbb{Z}\}.$$

Theorem 3.4. *There is a positive number δ^∞ such that if U is an open ball in a manifold M with radius smaller than δ^∞, then the set Λ_U contains infinitely many points.*

Proof. Let x_1 denote one of the homoclinic points through a periodic point (e.g. $x_1 \in W^s(y_1) \cap W^u(y_1)$ for y_1 a periodic point.) Then for any positive ϵ, there are finitely many points of $\mathcal{O}(x_1)$ whose distance from the origin is greater than ϵ. Let x_2 denote one of the homoclinic points through any other periodic point y_2 (where $\mathcal{O}(y_1) \cap \mathcal{O}(y_2) = \emptyset$). Then, again, for any positive ϵ, there are finitely many points of $\mathcal{O}(x_2)$ whose distance from $\mathcal{O}(y_2)$ is greater than ϵ.

Now, for every point $x \in M$, let d_1 be the distance between x and $\overline{\mathcal{O}(x_1)}$ and let d_2 be the distance between x and $\overline{\mathcal{O}(x_2)}$. Let r_x be the greater of d_1 and d_2. (Note that by definition, $\overline{\mathcal{O}(x_1)} \cap \overline{\mathcal{O}(x_2)} = \emptyset$, and both sets are closed, so $r_x > 0$.) Define V_x to be an open ball about the point x of radius r_x. Then the collection of all sets $\{V_x\}$ is an open cover for M. Hence there is some finite subcover of these sets. Let δ^∞ be the Lebesgue number for that finite subcover.

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Notice now that if $U \subset M$ is an open ball of radius smaller than δ^{∞}, then $U \subset V_x$ for some $x \in M$. Then either $\mathcal{O}(x_1)$ or $\mathcal{O}(x_2)$ does not intersect U by construction of each set V_x. So Λ_U contains a non-periodic point, and so Λ_U has infinite cardinality. □

A related theorem is as follows:

Theorem 3.5. *There is a positive number δ^{finite} such that if U is an open ball with radius greater than δ^{finite}, then the set Λ_U is finite.*

The proof of this depends on two preliminary results:

Lemma 3.6. *For a closed (compact) set K, define $d_K(x) = d(x, K)$, where $d(x, K)$ represents the distance function between x and K. Then $|d_K(x) - d_K(y)| \leq d(x, y)$.*

Proof. By the triangle inequality, we have

$$d(x, K) \leq d(x, y) + d(y, K)$$

and consequently

$$d(x, K) - d(y, K) \leq d(x, y)$$

or

$$d(y, K) - d(x, K) \geq -d(x, y).$$

Similarly, since

$$d(y, K) \leq d(x, y) + d(x, K)$$

we conclude

$$d(y, K) - d(x, K) \leq d(x, y).$$

Combining, we get

$$-d(x, y) \leq d(y, K) - d(x, K) \leq d(x, y)$$

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or

$$|d(x,K) - d(y,K)| = |d_K(x) - d_K(y)| \leq d(x,y)$$

as claimed. $\square$

Lemma 3.7. *Let J be an arbitrary collection of closed subsets of M. Define $g(x) = sup_{K \in J} d_K(x)$. Then $g(x)$ is continuous.*

Proof. Let $\epsilon > 0$ be given, and let x be an arbitrary point of M. Then for all y with $d(x,y) < \epsilon/2$, we know that $|d_K(x) - d_K(y)| < d(x,y)$ for all $K \in J$ by Lemma 3.6. Then by definition of $g(x)$, $|g(x) - g(y)| \leq \epsilon/2 < \epsilon$. Hence $g(x)$ is continuous. $\square$

We now give the proof of Theorem 3.5.

Proof. Let S be a finite set of periodic points of M together with every point in their respective orbits. For every point $x \notin S$, define $K_x = \overline{\mathcal{O}(x)}$, the closure of the orbit of x. Let J be the set of all K_x. Define the function $g(x)$ as in Lemma 3.7. Let δ^{finite} be the maximum of $g(x)$ on $\mathbb{T}^2$. Then for any point $x \in M$, if U is an open ball of radius greater than δ^{finite}, then U contains at least one point of the orbit of each point not in S. Hence, Λ_U is a subset of S, and must therefore be finite. $\square$

Remark: For both Theorems 3.4 and 3.5, no metric was specified for the distance functions giving the size of the open sets and the point-set distances. Both theorems proceed without difficulty if any complete metric is used. It is also easy to see that a complete metric is necessary, as the discrete metric readily gives counterexamples in both cases.

Remark: Theorem 3.5 holds for the extension of the problem into dimension higher than 2. Theorem 3.4 can also be extended by using two periodic points y_1 and y_2 by a similar procedure, although explicitly locating the homoclinic points is more difficult.

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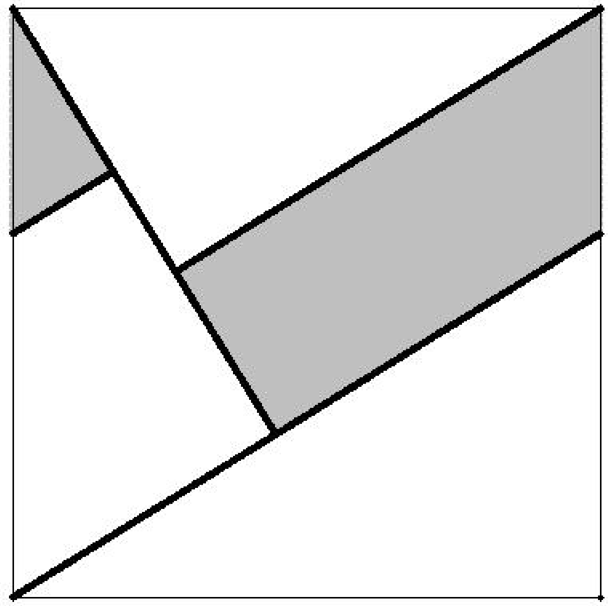

Figure 3.1: A partition of $\mathbb{T}^2$ with two rectangles.

A specific computation of δ^{finite} and δ^{∞} can be somewhat difficult, if not impossible. However, in the case of hyperbolic toral automorphisms we can establish some bounds for these values.

Example 3.8. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consider the Markov partition of $\mathbb{T}^2$ given in Figure 3.1. Let R_1 be the larger, white rectangle and R_2 be the shaded rectangle. Then the transition matrix for this example turns out to be

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

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just the same as A. Now, let U be the interior of R_1. As can be seen by the transition matrix, removing the interior of R_1 leaves no permissible transitions in the set Λ_{R_1} from the interior of one rectangle to the interior of another. Hence, the only points of Λ_{R_1} are the homoclinic points through the origin and the origin itself. (Note that a point whose orbit lands in the interior of R_2 exactly once and then lies on $W^s(0)$ or $W^u(0)$ would also be permissible, but in this case, no such point exists.) In any event, Λ_{R_1} has infinitely many points, so δ^{finite} needs to be large enough so as to enclose all of R_1. Again, the particular value will depend on the metric used.

The set Λ_{R_1} given in this example is countably infinite, as the orbit of each homoclinic point is countable, and there are only finitely many of them whose orbits never fall into R_1, namely, the points at the corners of the rectangles in the partition. Uncountable sets Λ_U are also possible. In fact, if we let U be a subset of the interior of R_1 so that every point of U is at least some fixed distance from the boundary of R_1, then the Smale-Birkhoff theorem (see background) applies and we obtain (among other things) Λ_U being uncountable. On the other hand, if the set U is expanded so as to include all of the boundary of R_1, then no homoclinic orbits remain in Λ_U, so Λ_U is empty. Hence, Λ_U may have finite, countable, or uncountable cardinality.

One final topological/dynamical property of Λ_U can also be given that is relevant for our analysis later. Recall that an invariant set Λ is *locally maximal* if it contains a local product structure. Λ_{R_2} is not locally maximal, as can be demonstrated by Figure 3.2. Here, the thick points highlight the orbit of a homoclinic point, the thick lines are the boundaries of R_2, and the thinner lines are the (un)stable manifolds of the points in the orbit of the homoclinic point. Since the intersection of the stable and unstable manifolds of points near the origin all lie within R_2, they are not included in Λ_{R_2} by definition. Hence, Λ_{R_2} is not

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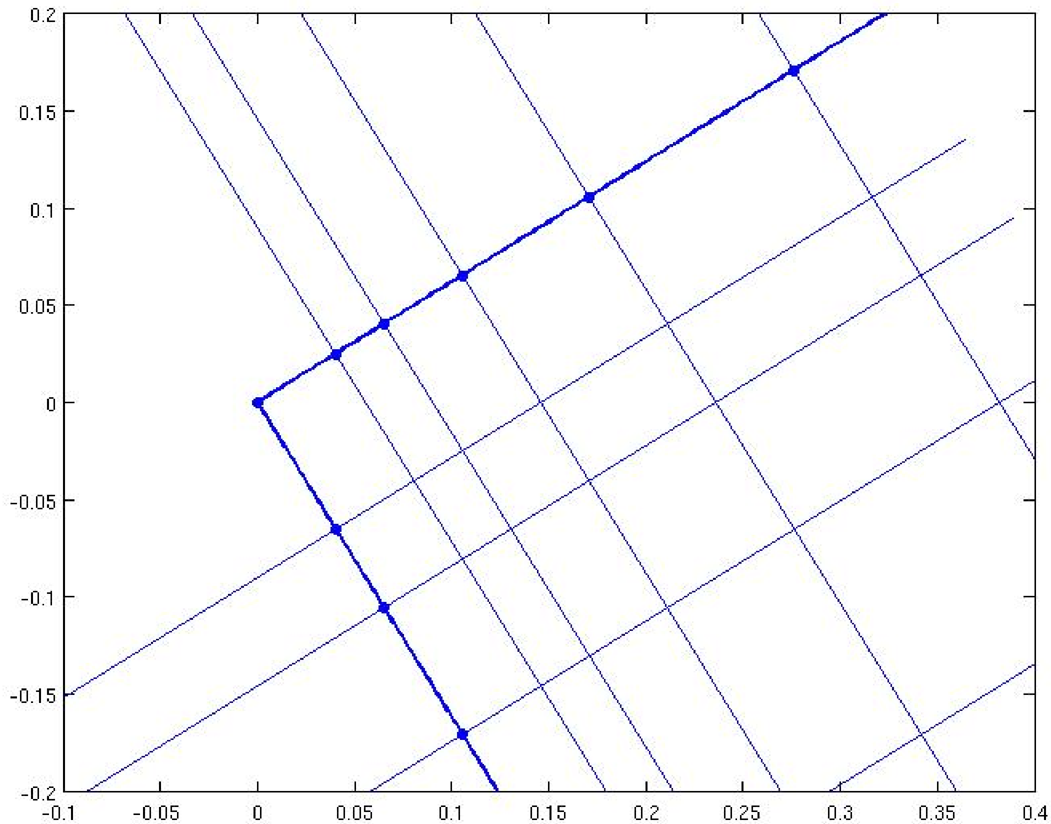

Figure 3.2: An example to show that Λ_U need not be locally maximal. Some points of Λ_U with (un)stable manifolds are drawn, showing the absence of a local product structure.

locally maximal. This argument is readily extended to any set Λ_U, where Λ_U represents the interior of a rectangle with one corner at the origin (or any other fixed point).

In this chapter, we have given some basic properties about the invariant sets Λ_U relating to connectivity and cardinality. Later, we will proceed to give some more "advanced" properties of Λ_U, with our ultimate goal being a determination of the Hausdorff dimension of the set. This will require a few more advanced tools and techniques.

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In the previous chapter, we gave some simple topological results of the invariant sets Λ_U. Here, we wish to develop a more advanced concept, that of the *Hausdorff dimension* of Λ_U. Among other things, the Hausdorff dimension will give a classification of the relative size of Λ_U, as Λ_U has Lebesgue measure zero for all open sets U.

Results for this chapter will be limited to the two-dimensional case. Additionally, we will require the set U to be simply connected.

4.1 Self-Similarity and Dimension

A highly desirable property of a fractal set is that of *self-similarity.* A set S is said to be *self-similar* if S can be subdivided into k congruent subsets, each of which may be magnified by a constant M such that the resulting set is identical to S. In this case, the *fractal dimension* d^{frac} of a self-similar set S is defined as

$$d^{\text{frac}}(S) = \frac{\log k}{\log M} = \log_M(k).$$

Some well-known examples of this definition are that of the Cantor middle-thirds set, which has dimension $\log_3(2)$, and the Sierpinski triangle, which has dimension $\log_2(3)$ (as in the introduction). It is well-known that the Hausdorff dimension of both of these sets are equal to their fractal dimension. For now, we will use the fractal dimension of Λ_U as an *a priori* estimate of the Hausdorff dimension. As an example, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

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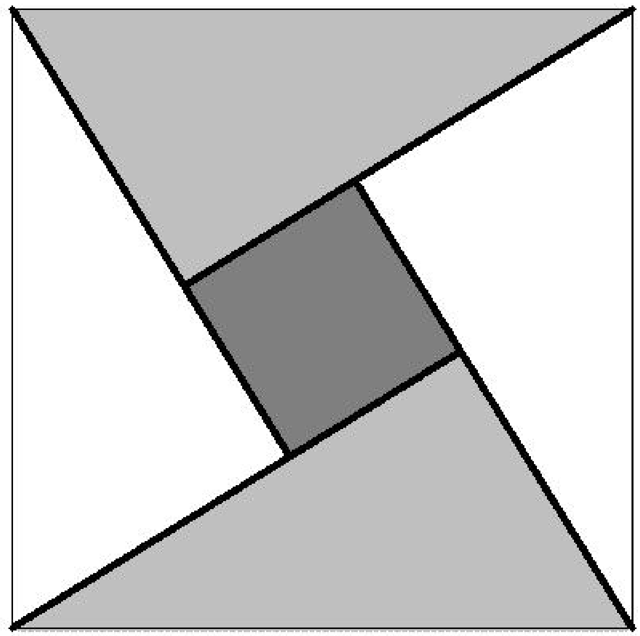

Figure 4.1: A Markov partition of $\mathbb{T}^2$. For the examples referencing this partition, R_1 is the white rectangle, R_2 is the lighter gray rectangle, and R_3 the darker gray rectangle.

and let U be the interior of R_3 in the Markov partition given in Figure 4.1.

Again, with the aid of a computer program, we can demonstrate the effect of performing a few iterations of the map. The results are given in Figure 4.2. The self-similarity of the image is readily apparent, and it is not difficult to see that continuing in a similar fashion will yield a set Λ_U that resembles Cantor dust (a cross product of the Cantor set with itself). In this case, because of the similarities present, we can use some simple algebra and geometry to compute $d^{\mathrm{frac}}(\Lambda_U)$. Doing so, we obtain $\log_\varphi(2)$, or approximately 1.44042009. Here, $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio. This is also the larger of the two eigenvalues of A.

The fractal dimension of Λ will ultimately depend on two things: The set that is removed

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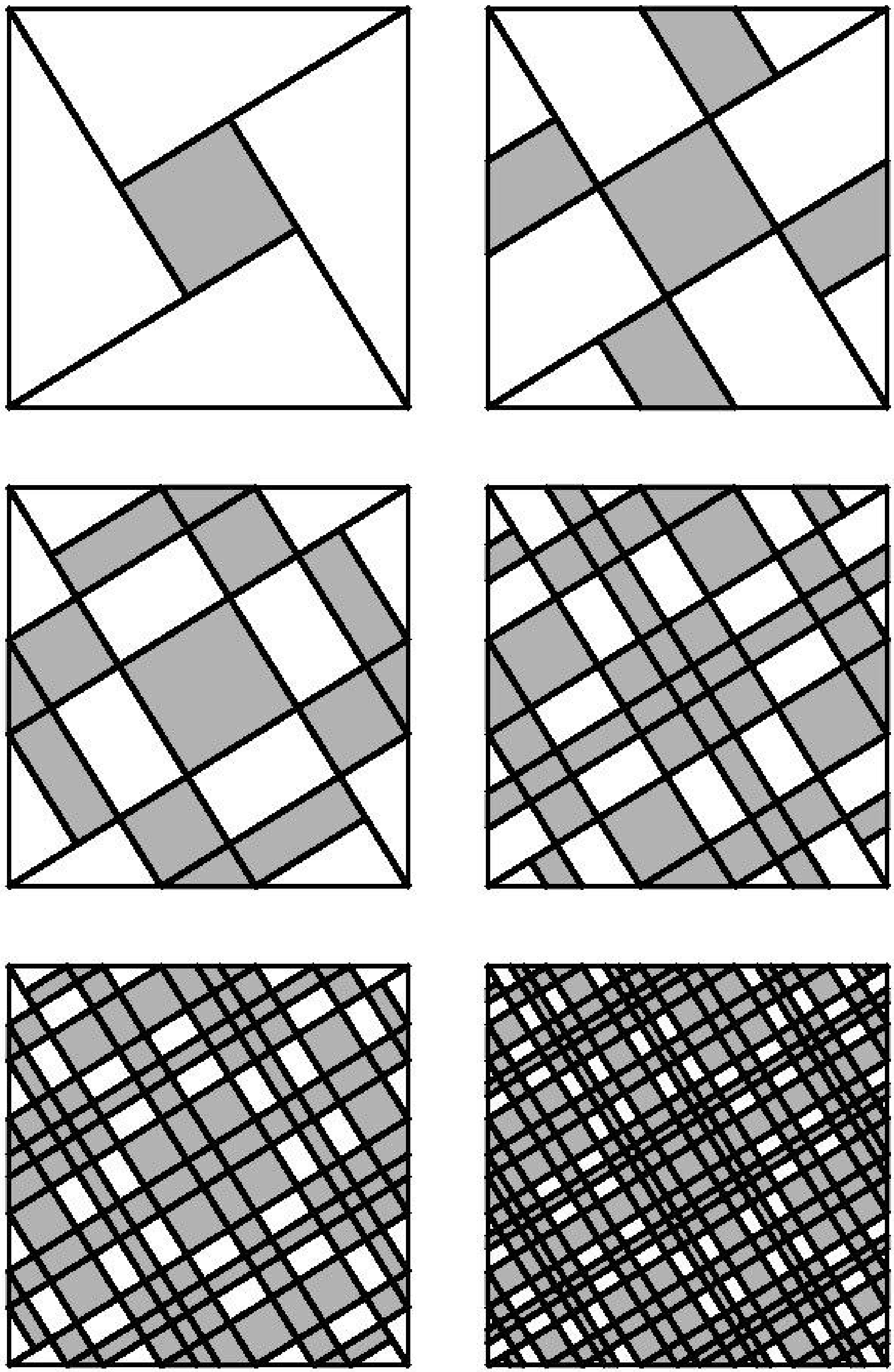

Figure 4.2: Successive approximations of the set Λ_U for the map f_A. The shaded region represents $\cup_{i=-m}^{i=m} f_A^i(U)$ for $m = 0, 1, ...5$ (left to right, top to bottom).

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from $\mathbb{T}^2$, and the matrix A. Consider now the effect of using the map

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvectors are in the same direction, so we can begin with the same partition. However, even if R_3 is used, a pattern is not so obvious (see Figure 4.3). Hence, we need some more sophisticated machinery to approach this problem.

4.2 A Connection between Entropy and Dimension

The following theorem gives a method with which we can compute the box dimension of the invariant sets Λ_U. It is based on a result of Fathi (see [20]), modified to work in the particular case of the two-dimensional hyperbolic toral automorphism.

Theorem 4.1. *Let f_A be a two-dimensional hyperbolic toral automorphism, and let λ be the absolute value of the eigenvalue of A greater than 1. Then if K is a compact f_A-invariant set, the box dimension of K is given by*

$$\frac{2h_{top}(f_A|_K)}{\log \lambda}.$$

Proof. Let the metric d be given by the ∞-metric in terms of the basis given by the two eigenvectors of A. In this metric, a "ball" is a square with sides parallel to the eigenvectors of A. For instance, the interior of R_3 in Figure 4.1 is a ball about the point $(1/2, 1/2)$.

Let $\epsilon > 0$ such that $1/\lambda + \epsilon < 1$. If $B(x, \epsilon)$ represents the open ball about x of radius ϵ

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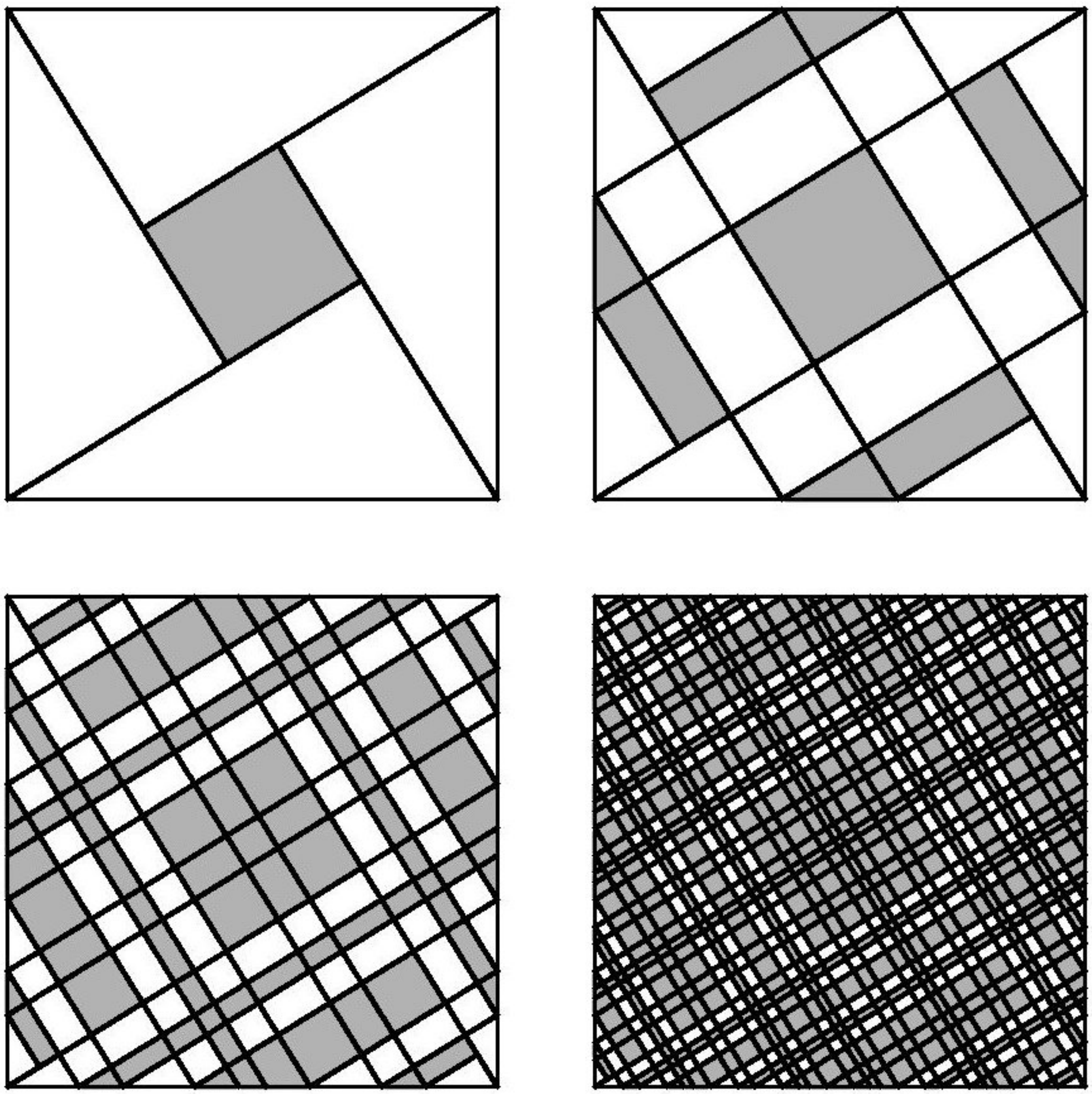

Figure 4.3: Successive approximations of the set Λ_U for the map f_{A^2}. The shaded region represents $\cup_{i=-m}^{i=m} f_{A^2}^i(U)$ for $m = 0, 1, 2, 3$ (left to right, top to bottom).

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in the metric d, then for $n \in \mathbb{N}$ define $\mathcal{B}_n(x,\epsilon)$ to be

$$\mathcal{B}_n(x,\epsilon) = \bigcap_{i=-n}^{n} f_A^{-i} B(f_A^i(x),\epsilon).$$

Note that this is precisely equal to $B(x,\epsilon\lambda^{-n})$ by our choice of metric. Define $\mathcal{N}(n,\epsilon)$ to be the minimum number of sets $\mathcal{B}_n(x,\epsilon)$ required to cover K.

Lemma 4.2. $\mathcal{N}(n,\epsilon) \leq r_{span}(2n+1,\epsilon)$.

Proof. Let Y be a minimal $(2n+1,\epsilon)$-spanning set for K. Then for all $x \in K$, we have that there is a point y of Y such that $d(f^i(x), f^i(y)) < \epsilon$ for all $0 \leq i \leq 2n+1$. But then it certainly holds that $d(f^i(x), f^i(y)) < \epsilon$ for all $0 \leq i \leq 2n$, and so Y is a $(2n,\epsilon)$-spanning set for K as well.

Now, consider the set $\{\mathcal{B}_n(f^n(y),\epsilon) \ : \ y \in Y\}$. We claim that this forms a cover for the set K. To see this, note that

$$\begin{aligned}\mathcal{B}_n(f^n(y),\epsilon) &= \{x \in K : d(f^i(f^n(y)), f^i(x)) < \epsilon \ : \ -n \leq i \leq n\} \\ &= \{x \in K : d(f^i(y)), f^i(x)) < \epsilon \ : \ 0 \leq i \leq 2n\}\end{aligned}$$

Since Y is a $(2n,\epsilon)$-spanning set, every point $x \in K$ is included in one of the sets $\mathcal{B}_n(f^n(y),\epsilon)$. Hence, the collection $\{\mathcal{B}_n(f^n(y),\epsilon) \ : \ y \in Y\}$ is a cover for K. As this cover may not be minimal, we get $\mathcal{N}(n,\epsilon) \leq r_{\text{span}}(2n+1,\epsilon)$.

□

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We now return to the proof of Theorem 4.1. By definition, we have that

$$\begin{aligned} h_{\text{top}}(f_A|_K) &= \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{\log(r_{\text{span}}(2n+1,\epsilon))}{2n+1}\right) \\ &\geq \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log(\mathcal{N}(n,\epsilon))}{2n+1} \end{aligned}$$

with the inequality following from Lemma 4.2. Multiplying both sides by 2 yields

$$2h_{\text{top}}(f_A|_K) \geq \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log(\mathcal{N}(n,\epsilon))}{n+\frac{1}{2}} = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log(\mathcal{N}(n,\epsilon))}{n}.$$

Let $N(\epsilon, K)$ be the minimum number of balls of radius ϵ that are needed to cover K, as in the definitions for upper and lower box dimensions. Recall that the definition of upper box dimension is

$$\dim_B(K) = \limsup_{\epsilon\to 0}\frac{\log(N(\epsilon,K))}{-\log(\epsilon)}.$$

Since $\mathcal{N}(n,\epsilon)$ measures the number of balls required to cover K each having radius $\epsilon\lambda^{-n}$, we can also write the box dimension in terms of these quantities as follows:

$$\dim_B(K) = \frac{\log(\mathcal{N}(n,\epsilon))}{-\log(\epsilon\lambda^{-n})} = \frac{\log(\mathcal{N}(n,\epsilon))}{-\log(\epsilon)+n\log(\lambda)} = \frac{\frac{\log(\mathcal{N}(n,\epsilon))}{n}}{\frac{-\log(\epsilon)}{n}+\log(\lambda)}.$$

We then know that

$$\begin{aligned} \dim_B(K) &= \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\frac{\log(\mathcal{N}(n,\epsilon))}{n}}{\frac{-\log(\epsilon)}{n}+\log(\lambda)} \\ &= \lim_{\epsilon\to 0}\frac{\limsup_{n\to\infty}\frac{\log(\mathcal{N}(n,\epsilon))}{n}}{\log(\lambda)} \\ &= \frac{\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log(\mathcal{N}(n,\epsilon))}{n}}{\log(\lambda)} \\ &\leq \frac{2h_{\text{top}}(f_A|_K)}{\log(\lambda)}, \end{aligned}$$

which establishes an upper bound for the box dimension.

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On the other hand, we have the following lemma, which will aid in obtaining a lower bound for the box dimension:

Lemma 4.3. *For sufficiently large n (depending only on λ), $\mathcal{N}(n, \epsilon) \geq r_{sep}(2n-1, \epsilon)$.*

Proof. Let S be a maximal $(2n-1, \epsilon)$-separated set for K. Let U be the set $f_A^{-n}(\mathcal{B}_n(x, \epsilon))$ for some point x. Suppose there are two points $y, z \in U \cap S$. By construction of $\mathcal{B}_n(x, \epsilon)$, this means that both y and z remain within ϵ distance of x for $2n-1$ iterations of f_A. Then in the set U, it must be that the distance between x and y is less than $\epsilon\lambda^{-(2n-1)}$ and that the distance between x and z is less than $\epsilon\lambda^{-(2n-1)}$. For sufficiently large n, $\lambda^{-(2n-1)}$ is less than $1/2$, and so by the triangle inequality, we get that y and z are within ϵ of each other, violating the assumption that S was $(2n-1, \epsilon)$-separated. Hence, each set U can contain at most one point of S. □

We now finish the proof of Theorem 4.1. Lemma 4.3 allows us to compute

$$\begin{aligned} h_{\text{top}}(f_A|_K) &= \lim_{\epsilon \to 0} \left(\liminf_{n \to \infty} \frac{\log(r_{\text{sep}}(2n-1, \epsilon))}{2n-1} \right) \\ &\leq \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log(\mathcal{N}(n, \epsilon))}{2n-1} \end{aligned}$$

for sufficiently large n. This again results in

$$2h_{\text{top}}(f_A|_K) \leq \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log(\mathcal{N}(n, \epsilon))}{n - \frac{1}{2}} = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log(\mathcal{N}(n, \epsilon))}{n}$$

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when both sides are multiplied by 2. We then compute the inner box dimension to be

$$\begin{aligned}
\dim_b(K) &= \lim_{\epsilon\to 0}\liminf_{n\to\infty} \frac{\frac{\log(\mathcal{N}(n,\epsilon))}{n}}{\frac{-\log(\epsilon)}{n} + \frac{n-1}{n}\log(\lambda)} \\
&= \lim_{\epsilon\to 0} \frac{\liminf_{n\to\infty}\frac{\log(\mathcal{N}(n,\epsilon))}{n}}{\log(\lambda)} \\
&= \frac{\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log(\mathcal{N}(n,\epsilon))}{n}}{\log(\lambda)} \\
&\geq \frac{2h_{\text{top}}(f_A|_K)}{\log(\lambda)},
\end{aligned}$$

which establishes the lower bound for the box dimension and completes the proof. □

As an example, consider again the mapping f_A where A is the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the Markov partition in the example from earlier, we obtain the transition matrix

$$T_A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

and by removing R_3, we end up with a smaller transition matrix

$$T_A^* = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Some rudimentary linear algebra tells us that the eigenvalues (hence entropy) of T_A^* are the same as the eigenvalues of the upper-left 2×2 block of T_A and zero. In fact, in order to compute the transition matrix given by removing one rectangle, we simply remove the row

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and column corresponding to that rectangle. Additionally, we can then remove any rectangle whose column in the transition matrix contains all zeros (as there are no points whose orbits fall into such a rectangle) or whose row in the transition matrix contains all zeros (as the orbit of any such point has nowhere to go). Repeating this eventually yields a smaller matrix with the same non-zero eigenvalues.

For verification, we compare this result against our estimate of the dimension we obtained in the previous chapter. We find that the eigenvalues of T_A^* are $\pm\sqrt{2}$, so $\lambda = \log(\sqrt{2})$, and we obtain $2\log(\sqrt{2})/\log(\varphi) = \log(2)/\log(\varphi)$ as the box dimension.

If the same partition is used for the matrix A^2 and the same rectangle is removed, the entropy obtained is 4 and the value of λ is φ^2. Hence, the dimension of the Λ_{R^3} in Figure 4.3 is the same as in Figure 4.2, namely $\log(2)/\log(\varphi)$.

Remark: A result by Pesin (see [16], Theorem 22.2) gives the Hausdorff dimension of a locally maximal hyperbolic invariant set in terms of the entropy and the expanding and contracting constants of the map.

Theorem 4.4 (Pesin). *For a locally maximal hyperbolic set Λ of a $C^{1+\alpha}$ diffeomorphism which is both u- and s-conformal and topologically mixing, we have*

$$\dim_H(\Lambda) = \dim_b(\Lambda) = \dim_B(\Lambda) = t^{(u)} + t^{(s)}$$

where $t^{(u)}$ and $t^{(s)}$ are the unique roots of Bowen's equations

$$P_\Lambda(-t\log|a^{(u)}|) = 0, \quad P_\Lambda(t\log|a^{(s)}|) = 0$$

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respectively and can be computed by the formulae

$$t^{(u)} = \frac{h_{K^{(u)}}(f)}{\int_\Lambda \log|a^{(u)}|(x)|d_{K^{(u)}}(x)},$$

$$t^{(s)} = -\frac{h_{K^{(s)}}(f)}{\int_\Lambda \log|a^{(s)}|(x)|d_{K^{(s)}}(x)},$$

where K is the unique equilibrium measure corresponding to the function $-t^{(u)}\log|a^{(u)}(x)|$ on Λ.

We refer the reader to the background chapter to review the definitions of conformality and Bowen's equation. In the context of two-dimensional hyperbolic toral automorphisms, $a^{(u)}(x)$ is the absolute value of the eigenvalue λ with modulus greater than one, and $a^{(s)}$ is $|\frac{1}{\lambda}|$. Further, the measure $K^{(u)}(f)$ is ergodic, so the value of the denominator is precisely $\log(\lambda)$, just as in Theorem 4.1. Moreover, the measure K will be determined by taking the supremum over all invariant measures μ of the function

$$h_\mu(f) - t\log(\lambda)$$

for fixed t. For two-dimensional hyperbolic toral automorphisms, this is precisely the measure corresponding to the topological entropy of the system. Hence, we obtain that the Hausdorff dimension of the sets Λ_U is precisely

$$\begin{aligned} t^{(u)} + t^{(s)} &= \frac{h_{\text{top}}(f_A|_{\Lambda_U})}{\log(\lambda)} - \frac{h_{\text{top}}(f_A|_{\Lambda_U})}{\log(\frac{1}{\lambda})} \\ &= \frac{h_{\text{top}}(f_A|_{\Lambda_U})}{\log(\lambda)} + \frac{h_{\text{top}}(f_A|_{\Lambda_U})}{\log(\lambda)} \\ &= \frac{2h_{\text{top}}(f_A|_{\Lambda_U})}{\log\lambda}, \end{aligned}$$

which is equal to the box dimension computed above.

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An additional result by Palis and Viana (see [21]) shows that if Λ is a locally maximal hyperbolic set of a C^1 surface diffeomorphism with stable and unstable dimension equal to 1, then there is a neighborhood $\mathcal{U}$ of f so that the Hausdorff dimension of Λ varies continuously as f varies within $\mathcal{U}$. Mañe (see [22]) strengthened this result by showing that if $r \geq 2$ and f is a C^r surface diffeomorphism, then the dimension varies in a C^{r-1} fashion.

Remark: It was necessary to modify Theorem 1.2 in [20] as the original theorem gives a crude general upper bound. As stated by Fathi, the theorem is as follows:

Theorem 4.5. *(Fathi) Let K be a compact subset of the manifold M. Suppose that K is hyperbolic for the C^1 diffeomorphism f. Define*

$$\lambda = \max\left[\lim_{n\to\infty}\frac{1}{n}\log\left(\max_{x\in K}||T_xf^n|E^s||\right), \lim_{n\to\infty}\frac{1}{n}\log\left(\max_{x\in K}||T_xf^{-n}|E^u||\right)\right],$$

then we have:

$$\dim_H(K) \leq \dim_B(K) \leq \frac{-2h_{top}(f|_K)}{\lambda}.$$

(Note here that λ corresponds to the logarithm of the eigenvalue of modulus smaller than one in the case of hyperbolic toral automorphism in two dimensions.) To demonstrate that equality does not necessarily hold, let A again be the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

let O be the 2×2 zero matrix, and consider the hyperbolic toral automorphism on $\mathbb{T}^4$ be given by the block matrix

$$B = \begin{bmatrix} A & O \\ O & A^n \end{bmatrix}$$

for arbitrary $n \in \mathbb{N}$. Then the value of λ (as presented in Fathi's theorem) is $1/\varphi$. Let K the

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subset $\mathbb{T}^2 \times \{[0]\}$ of $\mathbb{T}^4$, where $[0]$ is the projection of the origin in $\mathbb{R}^2$ onto the 2-torus. Note that this set is unaffected by the A^n submatrix of B. Further, this set is certainly invariant under f_B and has box dimension 2. However, by Fathi's argument, the box dimension of this set is bounded above by

$$\frac{-2h_{\text{top}}(f_B|_K)}{\lambda} = \frac{2\log(\varphi^n)}{\log(\varphi)} = 2n.$$

As n is arbitrary, this quantity can be made as large as desired.

4.3 Two Theorems from Lind

The following two theorems are due to Lind (see [23], Section 5). The first gives the change in the spectral radius of a matrix in terms of the change in an entry of that matrix.

Theorem 4.6. *Let $A = [a_{ij}]$ be a real square matrix with simple eigenvalue λ and corresponding left eigenvector v and right eigenvector w. Then*

$$\left[\frac{\partial \lambda}{\partial a_{ij}}\right] = \frac{wv}{vw}.$$

Note that the right side of this expression is actually a matrix. The change in λ resulting by a small change in the ij entry of A is given by the ij entry of wv/vw.

In terms of the transition matrices used with the Markov partitions for hyperbolic toral automorphisms, we know that w and v have all positive entries, so we get that the spectral radius (hence entropy) increases as the a_{ij} entry increases, and the spectral radius decreases as a_{ij} decreases. Applying this enables us to estimate the dimension of Λ_U for arbitrary open sets U.

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Theorem 4.7. *Let T represent a transition matrix for a subshift of finite type with λ the eigenvalue of maximum modulus. Let σ_T represent the subshift of finite type defined by T. Let B be a T-admissible word, and $\sigma_{T<B>}$ represent the subshift of finite type defined by T where the word B does not occur. If k is the length of B, then there are constants c and d so that*

$$c\lambda^{-k} < h_{top}(\sigma_T) - h_{top}(\sigma_{T<B>}) < d\lambda^{-k}.$$

As Markov partitions are refined in $\mathbb{T}^2$, the resulting smaller rectangles can be coded by words of increasing length, using words in terms of the original partition. As such, we obtain the following corollary:

Corollary 4.8. *Let $U_1 \supset U_2 \supset U_3 ...$ where each U_i is the interior of a rectangle in some refinement of a Markov partition, then the dimension of the sets $\Lambda_{U_1}, \Lambda_{U_2}, \Lambda_{U_3} ...$ approaches 2.*

4.4 Dimension from Arbitrary Holes

As an example, consider letting U be a small circular hole around the point $(1/2, 1/2)$, and let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consider the Markov partition of 355 rectangles given in Figure 4.4. Here, the darker colored rectangles are entirely contained within the circle U, and the lighter colored rectangles are rectangles that overlap U. Using a computer program, we can compute two different box dimensions: the box dimension resulting from removing all of the shaded rectangles and the box dimension resulting from removing only the darker shaded rectangles. In Figure 4.4, the two entropies are given by log(1.481631) and log(1.568985) respectively. This gives the box

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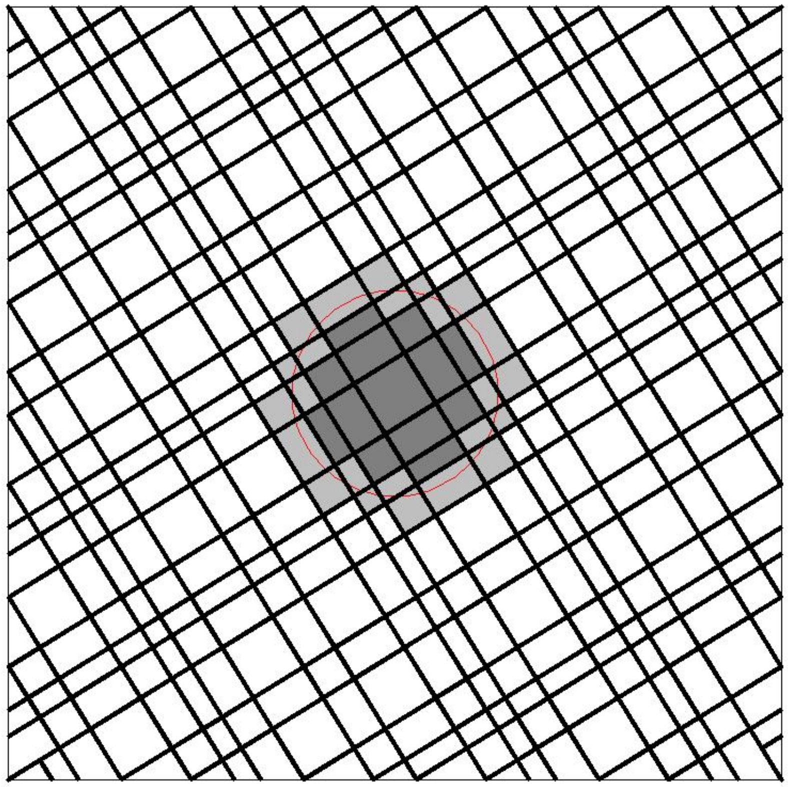

Figure 4.4: An example of using a Markov partition to approximate a small circular hole.

dimension of Λ_U between 1.634 and 1.872.

This estimate can be sharpened by refining the partition. In Figure 4.5, we refine to get a Markov partition containing 2439 rectangles. Here, the entropies are given by log(1.535355) and log(1.557403), yielding the box dimension between 1.782 and 1.841.

Continuing to refine the partitions, we can expect to converge to the box dimension. (A further refinement of 4.5 was attempted, but the computer ran out of memory.) In particular, since the refinements give better and better approximations to U, we can get an

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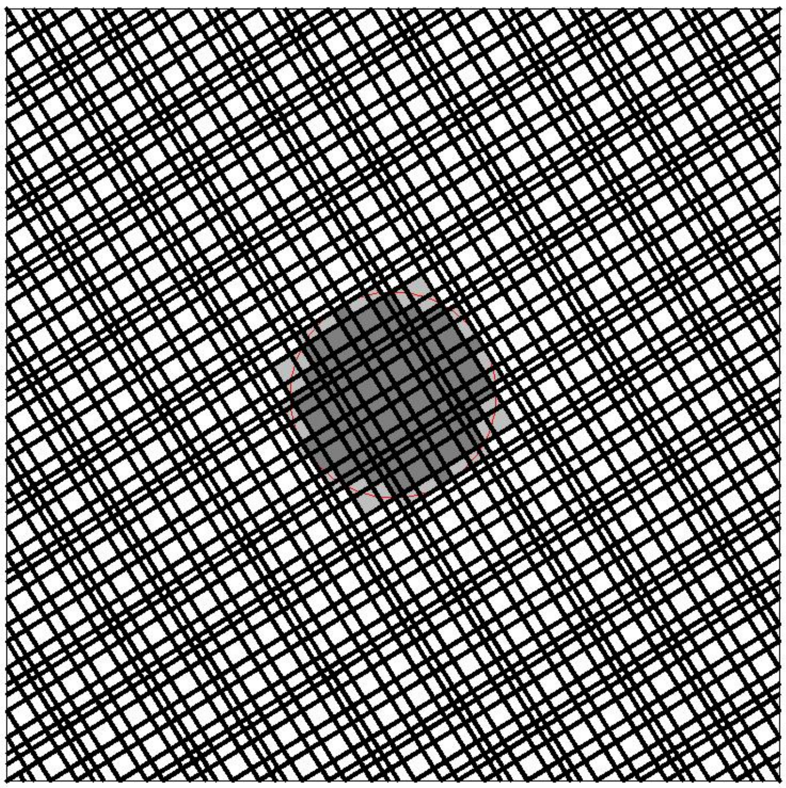

Figure 4.5: A refinement of the Markov partition from Figure 4.4.

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arbitrarily accurate estimate of the entropy of f_A on Λ_U, and, as a consequence, the box dimension. This is similar to the result by Lind in [23] in section 5 Theorem 3, in which the difference in entropy of a symbolic system with a word removed is computed, and shown to be exponentially decreasing in the length of the removed word. Further, this technique of moving from rectangles to arbitrary holes is analogous to one used in a series of papers by Chernov, Markarian, and Troubetzkoy (see for example [24] and [25]). In this series of papers, the authors consider ergodic measures for Anosov systems with holes. They are able to show rates of convergence to a fixed measure in terms of the size and placement of the holes on the manifold, as well as the particular map chosen.

Numerically, we have evidence for exponential convergence of the lower and upper entropy estimates to some fixed quantity as the Markov partition continues to be refined. This would then result in being able to compute the box dimension of the set Λ_U for arbitrary U. A combinatorial variation on Lind's theorem (Theorem 4.7) should give this analytically. Specifically, we need to demonstrate that the number of rectangles needed to approximate the open set U grows at such a rate so that the exponential decay of the entropy is still the dominating term in the limit. Time did not allow for this analysis to be performed.

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Appendix A. MATLAB Source Code

This is the MATLAB source code for program used to obtain the adjacency matrices for the examples of hyperbolic toral automorphisms. Some minor modifications to this program were used to draw all the images and make all the computations in this document.

```
%Helps to determine adjacency matrices for Markov partitions


%Inputs:
%   res: size of lattice
%   A: matrix to iterate
%   lens: Lengths of lines to draw Markov partitions
%   lpowr: lower power of iteration of Markov partition
%   upowr: upper power of iteration of Markov partition
%Output:
%   adjmat: The adjacency matrix of 1 iteration of the map


function adjmat = HTAadj(res,A,lens,lpowr,upowr)


%Define a matrix of colors for the drawings:


cvec = [1 0 0;
        0 1 0;
        0 0 1;
        1 1 0;
        0 1 1;
```


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```
        1 0 1;
        .5 0 0;
        0 .5 0;
        0 0 .5;
        .5 .5 0;
        0 .5 .5;
        .5 0 .5];

%first, draw the partition
disp('Drawing Partition Boundary');
pause(.1);

[V D] = eig(A);
lensl = zeros(2);
lensu = zeros(2);

lensl(1,:) = lens(1,:) * abs(D(1,1))^lpowr;
lensl(2,:) = lens(2,:) * abs(D(2,2))^lpowr;
lensu(1,:) = lens(1,:) * abs(D(1,1))^upowr;
lensu(2,:) = lens(2,:) * abs(D(2,2))^upowr;

points = zeros(1,2);
points = [points; getPoints(res,A,lensl)];
points = [points; getPoints(res,A,lensu)];

clf;
plot(points(:,1),points(:,2),'.','Color',[0 0 0]);
```


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```
axis square;

%next, determine what the partition on the grid will look like

disp('Defining rectangles');
pause(.1);
[pgrid, rect] = getPartition(res,points);

%draw this partition

disp('Coloring Rectangles');
pause(.1);
for s = 1:rect
    ptsofcolor = 0;
    for i=1:res
        for j=1:res
            if(pgrid(i,j)==s)
                ptsofcolor = ptsofcolor + 1;
            end;
        end;
    end;
    pts = zeros(ptsofcolor,2);
    idx = 1;
    for i=1:res
        for j=1:res
            if(pgrid(i,j)==s)
                pts(idx,:) = [i-1,j-1];
```


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```
                idx = idx + 1;
            end;
        end;
    end;
    colr = mod(s,size(cvec,1));
    if(colr == 0)
        colr = size(cvec,1);
    end;
    hold on;
    plot(pts(:,1),pts(:,2),'.','Color',cvec(colr,:));
end;
%Put the black points on over top
plot(points(:,1),points(:,2),'.','Color',[0 0 0]);

%Next, determine the adjacency matrix

adjmat = zeros(1);
pgrid2 = pgrid;

disp('Determining adjacency matrix');
pause(.1);
for i=0:res-1
    for j=0:res-1
        p1 = pgrid(i+1,j+1);
        x = A*[i;j];
        x = mod(x,res);
        p2 = pgrid(x(1)+1,x(2)+1);
```


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```
        if(p1 * p2 ~= 0)
            adjmat(p1,p2) = 1;
        end;
        pgrid2(i+1,j+1) = pgrid(x(1)+1,x(2)+1);
    end;
end;

disp(adjmat);

%Put the black points on over top
plot(points(:,1),points(:,2),'.','Color',[0 0 0]);

axis square;

function points = getPoints(res,A,lens)
%Inputs: As parent function
%Outputs: The points that give the Markov partition lines drawn

points = [0 0];

[V,D] = eig(A);
eval1 = V(:,1)';
eval2 = V(:,2)';
eval1 = eval1 / norm(eval1);
eval2 = eval2 / norm(eval2);

for t=lens(1,1):min(1/res, .001):lens(1,2)
```


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```
    points = [points; t*eval1];
end;

for t=lens(2,1):min(1/res, .001):lens(2,2)
    points = [points; t*eval2];
end;

points = mod(points * res, res);
points = [points; 0 res; res res; res 0];
return;

function [pgrid, rect] = getPartition(res, points)
%Inputs:
%   res: As in parent function
%   points: A list of points that are black on the map
%Outputs:
%   pgrid: A res-by-res matrix of integers determining which partition each point
%   is in, with the convention that the point at (x,y) corresponds to the
%   index (x+1, y+1) in the matrix
%   rects: the total number of rectangles in the partition

%This is the automated version of the function that will make searching for
%rectangles by hand obsolete

pgrid = zeros(res);

%first, black out all the relevant points (along the boundary of
```


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```
%rectangles)

for i = 1:size(points, 1)
    x = points(i,1);
    y = points(i,2);
    xhigh = mod(ceil(x), res);
    xlow = mod(floor(x), res);
    yhigh = mod(ceil(y), res);
    ylow = mod(floor(y), res);
    pgrid(xlow+1, ylow+1) = -1;
    if(xhigh <= res)
        pgrid(xhigh+1, ylow+1) = -1;
    end;
    if(yhigh <= res)
        pgrid(xlow+1, yhigh+1) = -1;
    end;
    if(xhigh <= res && yhigh <= res)
        pgrid(xhigh+1, yhigh+1) = -1;
    end;
end;

%systematically iterate over each point and floodfill
rect = 0;
for i=1:res
    for j=1:res
        if(pgrid(i,j)==0)
            rect = rect + 1;
```


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```
            pgrid(i,j) = rect;
            pgrid = floodfill(pgrid,rect);
        end;
    end;
end;

%clear out the -1 points (set them to zero)
for i=1:res
    for j=1:res
        if(pgrid(i,j)==-1)
            pgrid(i,j) = 0;
        end;
    end;
end;

return;

function ngrid = floodfill(pgrid,colr)
%Inputs:
%   pgrid: the grid from getPartition
%   colr: the number of the color to iteratively floodfill
%Outputs:
%   ngrid: a grid that is floodfilled based on a given color

%Idea: Use least fixed-point algorithm to floodfill the rectangle

changed = 1;
```


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```
res = size(pgrid,1);
ngrid = pgrid;

while(changed == 1)
    changed = 0;        %reset at the start of each loop
    for i=1:res
        for j=1:res
            %if colr found, check neighbors and change as needed
            if(ngrid(i,j)==colr)
                left = i-1;
                if(left == 0);
                    left = res;
                end;
                right = i+1;
                if(right == res+1)
                    right = 1;
                end;
                up = j + 1;
                if(up == res+1)
                    up = 1;
                end;
                down = j - 1;
                if(down == 0)
                    down = res;
                end;
                if(ngrid(i,up) == 0)
                    ngrid(i,up) = colr;
```


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```
                changed = 1;
            end;
            if(ngrid(i,down) == 0)
                ngrid(i,down) = colr;
                changed = 1;
            end;
            if(ngrid(left,j) == 0)
                ngrid(left,j) = colr;
                changed = 1;
            end;
            if(ngrid(right,j) == 0)
                ngrid(right,j) = colr;
                changed = 1;
            end;
        end; %done changing color
    end; %iterate over j coordinate
end; %iterate over i coordinate
end; %end fixed point iteration
```


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